HARDY AND RELLICH INEQUALITIES FOR ANISOTROPIC p-SUB-LAPLACIANS

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ABSTRACT. In this paper we establish the subelliptic Picone type identities. As consequences, we obtain Hardy and Rellich type inequalities for anisotropic p-sub-Laplacians which are operators of the form

$$\mathcal{L}_p f := \sum_{i=1}^{N} X_i \left(|X_i f|^{p_i-2} X_i f\right), \quad 1 < p_i < \infty,$$

where X_i, $i = 1, \ldots, N$, are the generators of the first stratum of a stratified (Lie) group. Moreover, analogues of Hardy type inequalities with multiple singularities and many-particle Hardy type inequalities are obtained on stratified groups.

1. Introduction

1.1. Historical background.

Recall the anisotropic Laplacian on $\mathbb{R}^N$ for $p_i > 1$ where $i = 1, \ldots, N$ (see [5]), defined by

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right), \tag{1.1}$$

which has recently attracted considerable attention. For instance, by taking $p_i = 2$ or $p_i = p = const$ (see [4]) in (1.1) we get the Laplacian and the pseudo-p-Laplacian, respectively. The anisotropic Laplacian has the theoretical importance not only in mathematics, but also many practical applications in the natural sciences. There are several examples: it reflects anisotropic physical properties of some reinforced materials (Lions [8] and Tang [16]), as well as explains the dynamics of fluids in the anisotropic media when the conductivities of the media are different in each direction [2] and [3]. It has also applications in the image processing [17].

The main purpose of this paper is to obtain the subelliptic Hardy and Rellich type inequalities for the anisotropic p-sub-Laplacian on stratified groups. First, we derive the subelliptic Picone type identities on stratified groups. As consequences, Hardy and Rellich type inequalities for anisotropic sub-Laplacians are presented. These results are given in Sections 2 and 3. In Section 4 and 5, we present analogues of Hardy type inequalities with multiple singularities and

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many-particle Hardy type inequalities on stratified groups, respectively. These inequalities are obtained in their horizontal form in the spirit of [13] (see also [14]). In Section 6, Hardy type inequalities with exponential weights are shown.

1.2. **Preliminaries.** Let $\mathbb{G} = (\mathbb{R}^d, \circ, \delta_\lambda)$ be a stratified Lie group (or a homogeneous Carnot group), with dilation structure δ_λ and Jacobian generators $X_1, \ldots, X_N$, so that N is the dimension of the first stratum of $\mathbb{G}$. We denote by Q the homogeneous dimension of $\mathbb{G}$. We refer to [11, 12] and to the recent book [15] for extensive discussions of stratified Lie groups and their properties.

The sub-Laplacian on $\mathbb{G}$ is given by

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2. \tag{1.2}$$

We also recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [15]). The left invariant vector field X_j has an explicit form and satisfies the divergence theorem, see e.g. [15] for the derivation of the exact formula: more precisely, we can formulate

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', ..., x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \tag{1.3}$$

with $x = (x', x^{(2)}, \ldots, x^{(r)})$, where r is the step of $\mathbb{G}$ and $x^{(l)} = (x_1^{(l)}, \ldots, x_{N_l}^{(l)})$ are the variables in the l^{th} stratum and $a_{k,m}^{(l)}$ is a δ_λ-homogeneous polynomial function of degree $l-1$. The horizontal gradient is given by

$$\nabla_{\mathbb{G}} := (X_1, \ldots, X_N),$$

and the horizontal divergence is defined by

$$\mathrm{div}_{\mathbb{G}} v := \nabla_{\mathbb{G}} \cdot v.$$

The horizontal anisotropic p-sub-Laplacian is defined by

$$\mathcal{L}_p f := \sum_{i=1}^{N} X_i \left(|X_i f|^{p_i - 2} X_i f \right), \quad 1 < p_i < \infty, \tag{1.4}$$

and we use the notation

$$|x'| = \sqrt{x_1'^2 + \ldots + x_N'^2}$$

for the Euclidean norm on $\mathbb{R}^N$.

2. **Subelliptic anisotropic Hardy type inequality**

First, we obtain the subelliptic Picone type identity on a stratified group $\mathbb{G}$. Here we follow the method of Allegretto and Huang [1] for the (Euclidean) p-Laplacian (see also [10]).

Lemma 2.1. *Let $\Omega \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let u, v be differentiable a.e. in Ω, $v > 0$ a.e. in Ω and $u \geq 0$, and denote*

$$R(u,v) := \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} X_i \left(\frac{u^{p_i}}{v^{p_i-1}} \right) |X_i v|^{p_i-2} X_i v,$$

$$L(u,v) := \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} X_i v X_i u + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i},$$

where $p_i > 1$, $i = 1, \ldots, N$. Then

$$L(u,v) = R(u,v) \geq 0. \tag{2.1}$$

In addition, let Ω be connected, then we have $L(u,v) = 0$ a.e. in Ω if and only if $u = cv$ a.e. in Ω with a positive constant c.

Note that the Euclidean case of this lemma was obtained by Feng and Cui [5].

Proof of Lemma 2.1. A direct computation gives

$$\begin{aligned} R(u,v) &= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} X_i \left(\frac{u^{p_i}}{v^{p_i-1}} \right) |X_i v|^{p_i-2} X_i v \\ &= \sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} X_i v X_i u + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i} \\ &= L(u,v). \end{aligned}$$

This proves the equality in (2.1). Now we rewrite $L(u,v)$ to see $L(u,v) \geq 0$, that is,

$$\begin{aligned} L(u,v) = &\sum_{i=1}^{N} |X_i u|^{p_i} - \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-1} |X_i u| + \sum_{i=1}^{N} (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i v|^{p_i} \\ &+ \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} \left(|X_i v||X_i u| - X_i v X_i u \right) \\ =& S_1 + S_2, \end{aligned}$$

where we denote

$$\begin{aligned} S_1 := &\sum_{i=1}^{N} p_i \left[\frac{1}{p_i} |X_i u|^{p_i} + \frac{p_i - 1}{p_i} \left(\left(\frac{u}{v} |X_i v| \right)^{p_i - 1} \right)^{\frac{p_i}{p_i-1}} \right] \\ &- \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-1} |X_i u|, \end{aligned}$$

and

$$S_2 := \sum_{i=1}^{N} p_i \frac{u^{p_i-1}}{v^{p_i-1}} |X_i v|^{p_i-2} \left(|X_i v||X_i u| - X_i v X_i u\right).$$

We can see that $S_2 \geq 0$ due to $|X_i v||X_i u| \geq X_i v X_i u$. To check $S_1 \geq 0$ we need to use Young's inequality for $a \geq 0$ and $b \geq 0$

$$ab \leq \frac{a^{p_i}}{p_i} + \frac{b^{q_i}}{q_i}, \tag{2.2}$$

where $p_i > 1, q_i > 1$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1, \ldots, N$. It holds if and only if $a^{p_i} = b^{q_i}$, i.e. if $a = b^{\frac{1}{p_i-1}}$. Let us take $a = |X_i u|$ and $b = \left(\frac{u}{v}|X_i v|\right)^{p_i-1}$ in (2.2) to get

$$p_i |X_i u| \left(\frac{u}{v}|X_i v|\right)^{p_i-1} \leq p_i \left[\frac{1}{p_i}|X_i u|^{p_i} + \frac{p_i-1}{p_i}\left(\left(\frac{u}{v}|X_i v|\right)^{p_i-1}\right)^{\frac{p_i}{p_i-1}}\right]. \tag{2.3}$$

From this we see that $S_1 \geq 0$ which proves that $L(u,v) = S_1 + S_2 \geq 0$. It is easy to see that $u = cv$ implies $R(u,v) = 0$ since in the case $R(cv, v)$ both terms cancel each other out. Now let us prove that $L(u,v) = 0$ implies $u = cv$. Due to $u(x) \geq 0$ and $L(u,v)(x_0) = 0, \quad x_0 \in \Omega$, we consider the two cases $u(x_0) > 0$ and $u(x_0) = 0$.

(1) For the case $u(x_0) > 0$ we conclude from $L(u,v)(x_0) = 0$ that $S_1 = 0$ and $S_2 = 0$. Then $S_1 = 0$ implies

$$|X_i u| = \frac{u}{v}|X_i v|, \quad i = 1, \ldots, N, \tag{2.4}$$

and $S_2 = 0$ implies

$$|X_i v||X_i u| - X_i v X_i u = 0, \quad i = 1, \ldots, N. \tag{2.5}$$

The combination of (2.4) and (2.5) gives

$$\frac{X_i u}{X_i v} = \frac{u}{v} = c, \quad \text{with} \quad c \neq 0, \quad i = 1, \ldots, N. \tag{2.6}$$

(2) Let us denote $\Omega^* := \{x \in \Omega | u(x) = 0\}$. If $\Omega^* \neq \Omega$, then suppose that $x_0 \in \partial\Omega^*$. Then there exists a sequence $x_k \notin \Omega^*$ such that $x_k \to x_0$. In particular, $u(x_k) \neq 0$, and hence by the case 1 we have $u(x_k) = cv(x_k)$. Passing to the limit we get $u(x_0) = cv(x_0)$. Since $u(x_0) = 0, \quad v(x_0) \neq 0$, we get that $c = 0$. But then by the case 1 again, since $u = cv$ and $u \neq 0$ in $\Omega \backslash \Omega^*$, it is impossible that $c = 0$. This contradiction implies that $\Omega^* = \Omega$.

This completes the proof of Lemma 2.1. $\square$

As a consequence of the subelliptic Picone type identity, we present the Hardy type inequality for the anisotropic sub-Laplacian on $\mathbb{G}$. We recall that for $x \in \mathbb{G}$ we write $x = (x', x'')$, with x' being in the first stratum of $\mathbb{G}$.

Theorem 2.2. *Let $\Omega \subset \mathbb{G}\backslash\{x' = 0\}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Then we have*

$$\sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx \geq \sum_{i=1}^{N} \left(\frac{p_i - 1}{p_i}\right)^{p_i} \int_{\Omega} \frac{|u|^{p_i}}{|x_i'|^{p_i}} dx, \tag{2.7}$$

for all $u \in C^1(\Omega)$ and where $1 < p_i < N$ for $i = 1, \ldots, N$.

Before we start the proof of Theorem 2.2, let us establish the following Lemma 2.3.

Lemma 2.3. *Let $\Omega \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let constants $K_i > 0$ and functions $H_i(x)$ with $i = 1, \ldots, N$, be such that for an a.e. differentiable function v, such that $v > 0$ a.e. in Ω, we have*

$$-X_i(|X_i v|^{p_i - 2} X_i v) \geq K_i H_i(x) v^{p_i - 1}, \quad i = 1, \ldots, N. \tag{2.8}$$

Then, for all nonnegative functions $u \in C^1(\Omega)$ we have

$$\sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx \geq \sum_{i=1}^{N} K_i \int_{\Omega} H_i(x) u^{p_i} dx. \tag{2.9}$$

Proof of Lemma 2.3. In view of (2.1) and (2.8) we have

$$\begin{aligned}
0 \leq \int_{\Omega} L(u, v) dx &= \int_{\Omega} R(u, v) dx \\
&= \sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx - \sum_{i=1}^{N} \int_{\Omega} X_i \left(\frac{u^{p_i}}{v^{p_i - 1}}\right) |X_i v|^{p_i - 2} X_i v dx \\
&= \sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx + \sum_{i=1}^{N} \int_{\Omega} \frac{u^{p_i}}{v^{p_i - 1}} X_i \left(|X_i v|^{p_i - 2} X_i v\right) dx \\
&\leq \sum_{i=1}^{N} \int_{\Omega} |X_i u|^{p_i} dx - \sum_{i=1}^{N} K_i \int_{\Omega} H_i(x) u^{p_i} dx.
\end{aligned}$$

This completes the proof of Lemma 2.3. □

Proof of Theorem 2.2. Before using Lemma 2.3, we shall introduce the auxiliary function

$$v := \prod_{j=1}^{N} |x_j'|^{\alpha_j} = |x_i'|^{\alpha_i} V_i, \tag{2.10}$$

where $V_i = \prod_{j=1, j \neq i}^{N} |x_j'|^{\alpha_j}$ and $\alpha_j = \frac{p_j - 1}{p_j}$. Then we have

$$\begin{aligned}
X_i v &= \alpha_i V_i |x_i'|^{\alpha_i - 2} x_i', \\
|X_i v|^{p_i - 2} &= \alpha_i^{p_i - 2} V_i^{p_i - 2} |x_i'|^{\alpha_i p_i - 2\alpha_i - p_i + 2}, \\
|X_i v|^{p_i - 2} X_i v &= \alpha_i^{p_i - 1} V_i^{p_i - 1} |x_i'|^{\alpha_i p_i - \alpha_i - p_i} x_i'.
\end{aligned}$$

Consequently, we also have

$$-X_i(|X_iv|^{p_i-2}X_iv) = \left(\frac{p_i-1}{p_i}\right)^{p_i}\frac{v^{p_i-1}}{|x_i'|^{p_i}}. \tag{2.11}$$

To complete the proof of Theorem 2.2, we choose $K_i = \left(\frac{p_i-1}{p_i}\right)^{p_i}$ and $H_i(x) = \frac{1}{|x_i'|^{p_i}}$, and use Lemma 2.3. □

3. Subelliptic anisotropic Rellich type inequality

We now present the (second order) subelliptic Picone type identity. As a consequence, we obtain the Rellich type inequality for the anisotropic sub-Laplacian on $\mathbb{G}$.

Lemma 3.1. *Let $\Omega \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let u, v be twice differentiable a.e. in Ω and satisfying the following conditions: $u \geq 0$, $v > 0$, $X_i^2 v < 0$ a.e. in Ω for $p_i > 1$, $i = 1, \ldots, N$. Then we have*

$$L_1(u,v) = R_1(u,v) \geq 0, \tag{3.1}$$

where

$$R_1(u,v) := \sum_{i=1}^{N} |X_i^2 u|^{p_i} - \sum_{i=1}^{N} X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i^2 v|^{p_i-2}X_i^2 v,$$

and

$$\begin{aligned} L_1(u,v) := &\sum_{i=1}^{N} |X_i^2 u|^{p_i} - \sum_{i=1}^{N} p_i\left(\frac{u}{v}\right)^{p_i-1} X_i^2 u X_i^2 v |X_i^2 v|^{p_i-2} \\ &+ \sum_{i=1}^{N}(p_i-1)\left(\frac{u}{v}\right)^{p_i}|X_i^2 v|^{p_i} \\ &- \sum_{i=1}^{N} p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-1}}|X_i^2 v|^{p_i-2}X_i^2 v\left(X_i u - \frac{u}{v}X_i v\right)^2. \end{aligned}$$

Proof of Lemma 3.1. A direct computation gives

$$\begin{aligned} X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right) &= X_i\left(p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i u - (p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i v\right) \\ &= p_i(p_i-1)\frac{u^{p_i-2}}{v^{p_i-2}}\left(\frac{(X_i u)v - u(X_i v)}{v^2}\right)X_i u + p_i\frac{u^{p_i-1}}{v^{p_i-1}}X_i^2 u \\ &- p_i(p_i-1)\frac{u^{p_i-1}}{v^{p_i-1}}\left(\frac{(X_i u)v - u(X_i v)}{v^2}\right)X_i v - (p_i-1)\frac{u^{p_i}}{v^{p_i}}X_i^2 v \\ &= p_i(p_i-1)\left(\frac{u^{p_i-2}}{v^{p_i-1}}|X_i u|^2 - 2\frac{u^{p_i-1}}{v^{p_i}}X_i v X_i u + \frac{u^{p_i}}{v^{p_i+1}}|X_i v|^2\right) \end{aligned}$$

$$
\begin{aligned}
&+ p_i \frac{u^{p_i-1}}{v^{p_i-1}} X_i^2 u - (p_i - 1) \frac{u^{p_i}}{v^{p_i}} X_i^2 v \\
&= p_i(p_i-1) \frac{u^{p_i-2}}{v^{p_i-1}} \left(X_i u - \frac{u}{v} X_i v \right)^2 + p_i \frac{u^{p_i-1}}{v^{p_i-1}} X_i^2 u - (p_i-1) \frac{u^{p_i}}{v^{p_i}} X_i^2 v,
\end{aligned}
$$

which gives (3.1). By Young's inequality we have

$$
\frac{u^{p_i-1}}{v^{p_i-1}} X_i^2 u X_i^2 v |X_i^2 v|^{p_i-2} \leq \frac{|X_i^2 u|^{p_i}}{p_i} + \frac{1}{q_i} \frac{u^{p_i}}{v^{p_i}} |X_i^2 v|^{p_i}, \quad i = 1, \ldots, N,
$$

where $p_i > 1, q_i > 1$ $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Since $X_i^2 v < 0$ we arrive at

$$
\begin{aligned}
L_1(u,v) \geq & \sum_{i=1}^N |X_i^2 u|^{p_i} + \sum_{i=1}^N (p_i - 1) \frac{u^{p_i}}{v^{p_i}} |X_i^2 v|^{p_i} - \sum_{i=1}^N p_i \left(\frac{|X_i^2 u|^{p_i}}{p_i} + \frac{1}{q_i} \frac{u^{p_i}}{v^{p_i}} |X_i^2 v|^{p_i} \right) \\
& - \sum_{i=1}^N p_i(p_i-1) \frac{u^{p_i-2}}{v^{p_i-1}} |X_i^2 v|^{p_i-2} X_i^2 v \left| X_i u - \frac{u}{v} X_i v \right|^2 \\
& = \sum_{i=1}^N \left(p_i - 1 - \frac{p_i}{q_i} \right) \frac{u^{p_i}}{v^{p_i}} |X_i^2 v|^{p_i} \\
& - \sum_{i=1}^N p_i(p_i-1) \frac{u^{p_i-2}}{v^{p_i-1}} |X_i^2 v|^{p_i-2} X_i^2 v \left| X_i u - \frac{u}{v} X_i v \right|^2 \geq 0.
\end{aligned}
$$

This completes the proof of Lemma 3.1. □

Now we are ready to prove the subelliptic Rellich type inequality on $\mathbb{G}$.

Theorem 3.2. *Let $\Omega \subset \mathbb{G} \backslash \{x' = 0\}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Then for a function $u \geq 0$, $u \in C^2(\Omega)$, and $2 < \alpha_i < N - 2$ we have the following inequality*

$$
\sum_{i=1}^N \int_\Omega |X_i^2 u|^{p_i} dx \geq \sum_{i=1}^N C_i(\alpha_i, p_i) \int_\Omega \frac{|u|^{p_i}}{|x_i'|^{2p_i}} dx, \tag{3.2}
$$

where $1 < p_i < N$ for $i = 1, \ldots, N$, and

$$
C_i(\alpha_i, p_i) = (\alpha_i(\alpha_i - 1))^{p_i-1} (\alpha_i p_i - 2p_i - \alpha_i + 2)(\alpha_i p_i - 2p_i - \alpha_i + 1).
$$

Proof of Theorem 3.2. We introduce the auxiliary function

$$
v := \prod_{j=1}^N |x_j'|^{\alpha_j} = |x_i'|^{\alpha_i} V_i,
$$

we choose α_j later, and let $V_i = \prod_{j=1, j \neq i}^N |x_j'|^{\alpha_j}$. Then we have

$$
\begin{aligned}
X_i^2 v &= X_i(\alpha_i V_i |x_i'|^{\alpha_i - 2} x_i') = \alpha_i(\alpha_i - 1) V_i |x_i'|^{\alpha_i - 2}, \\
|X_i^2 v|^{p_i-2} &= (\alpha_i(\alpha_i-1))^{p_i-2} V_i^{p_i-2} |x_i'|^{\alpha_i p_i - 2p_i - 2\alpha_i + 4}, \\
|X_i^2 v|^{p_i-2} X_i^2 v &= (\alpha_i(\alpha_i-1))^{p_i-1} V_i^{p_i-1} |x_i'|^{\alpha_i p_i - 2p_i - \alpha_i + 2}.
\end{aligned}
$$

Consequently, we obtain

$$\begin{aligned}X_i^2(|X_i^2 v|^{p_i-2}X_i^2 v) =&(\alpha_i(\alpha_i-1))^{p_i-1}V_i^{p_i-1}X_i^2(|x_i'|^{\alpha_i p_i-2p_i-\alpha_i+2})\\ =&(\alpha_i(\alpha_i-1))^{p_i-1}(\alpha_i p_i-2p_i-\alpha_i+2)V_i^{p_i-1}X_i\left(|x_i'|^{\alpha_i p_i-2p_i-\alpha_i}x_i'\right)\\ =&(\alpha_i(\alpha_i-1))^{p_i-1}(\alpha_i p_i-2p_i-\alpha_i+2)(\alpha_i p_i-2p_i-\alpha_i+1)\\ &\times V_i^{p_i-1}|x_i'|^{\alpha_i(p_i-1)-2p_i}.\end{aligned}$$

Thus, for twice differentiable function $v>0$ a.e. in Ω with $X_i^2 v<0$ we have

$$X_i^2(|X_i^2|^{p_i-2}X_i^2 v)=C_i(\alpha_i,p_i)\frac{v^{p_i-1}}{|x_i'|^{2p_i}} \tag{3.3}$$

a.e. in Ω. Using (3.3) we compute

$$\begin{aligned}0\leq\int_\Omega L_1(u,v)dx=&\int_\Omega R_1(u,v)dx\\ =&\sum_{i=1}^N\int_\Omega |X_i^2 u|^{p_i}dx-\sum_{i=1}^N\int_\Omega X_i^2\left(\frac{u^{p_i}}{v^{p_i-1}}\right)|X_i^2 v|^{p_i-2}X_i^2 v dx\\ =&\sum_{i=1}^N\int_\Omega |X_i^2 u|^{p_i}dx-\sum_{i=1}^N\int_\Omega \frac{u^{p_i}}{v^{p_i-1}}X_i^2\left(|X_i^2 v|^{p_i-2}X_i^2 v\right)dx\\ =&\sum_{i=1}^N\int_\Omega |X_i^2 u|^{p_i}dx-\sum_{i=1}^N C_i(\alpha_i,p_i)\int_\Omega\frac{|u|^{p_i}}{|x_i'|^{2p_i}}dx.\end{aligned}$$

The proof of Theorem 3.2 is complete. ∎

4. Hardy type inequalities with multiple singularities on $\mathbb{G}$

In this section we obtain the analogue of the Hardy inequality with multiple singularities on a stratified group. The singularities are represented by a family $\{a_k\}_{k=1}^m\in\mathbb{G}$, where we write $a_k=(a_k',a_k'')$, with a_k' being in the first stratum of $\mathbb{G}$. We can also write $a_k'=(a_{k1}',\ldots,a_{kN}')$, so $(xa_k^{-1})'=x'-a_k'$.

Theorem 4.1. *Let $\Omega\subset\mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $N\geq 3$, $x=(x',x'')\in\mathbb{G}$ with $x'=(x_1',\ldots,x_N')$ being in the first stratum of $\mathbb{G}$, and let $a_k\in\mathbb{G}, k=1,\ldots,m$, be the singularities. Then we have*

$$\int_\Omega|\nabla_{\mathbb{G}}u|^2dx\geq\left(\frac{N-2}{2}\right)^2\int_\Omega\frac{\sum_{j=1}^N\left|\sum_{k=1}^m\frac{(xa_k^{-1})_j'}{|(xa_k^{-1})'|^N}\right|^2}{\left(\sum_{k=1}^m\frac{1}{|(xa_k^{-1})'|^{N-2}}\right)^2}|u|^2dx, \tag{4.1}$$

for all $u\in C_0^\infty(\Omega)$.

The Euclidean case of this inequality was obtained by Kapitanski and Laptev [7]. In (4.1), $(xa_k^{-1})_j'=x_j'-a_{kj}'$ denotes the j^{th} component of xa_k^{-1}.

Proof of Theorem 4.1. Let us introduce a vector-field $\mathcal{A}(x) = (\mathcal{A}_1(x), \dots, \mathcal{A}_N(x))$ to be specified later. Also let λ be a real parameter for optimisation. We start with the inequality

$$\begin{aligned} 0 &\le \int_\Omega \sum_{j=1}^{N} (|X_j u - \lambda \mathcal{A}_j u|^2) dx \\ &= \int_\Omega \left(|\nabla_{\mathbb{G}} u|^2 - 2\lambda \mathrm{Re} \sum_{j=1}^{N} \overline{\mathcal{A}_j u} X_j u + \lambda^2 \sum_{j=1}^{N} |\mathcal{A}_j|^2 |u|^2 \right) dx. \end{aligned}$$

By using the integration by parts we get

$$- \int_\Omega \left(\lambda^2 \sum_{j=1}^{N} |\mathcal{A}_j|^2 + \lambda \mathrm{div}_{\mathbb{G}} \mathcal{A} \right) |u|^2 dx \le \int_\Omega |\nabla_{\mathbb{G}} u|^2 dx. \tag{4.2}$$

We differentiate the integral on the left-hand side with respect to λ to optimise it, yielding

$$2\lambda |\mathcal{A}|^2 + \mathrm{div}_{\mathbb{G}} \mathcal{A} = 0,$$

for all $x \in \Omega$. This is a restriction on $\mathcal{A}(x)$ giving $\frac{\mathrm{div}_{\mathbb{G}} \mathcal{A}(x)}{|\mathcal{A}(x)|^2} = const$. For $\lambda = \frac{1}{2}$ we get

$$\mathrm{div}_{\mathbb{G}} \mathcal{A}(x) = -|\mathcal{A}(x)|^2. \tag{4.3}$$

Then putting (4.3) in (4.2) we have the following Hardy inequality

$$\frac{1}{4} \int_\Omega \sum_{j=1}^{N} |\mathcal{A}_j(x)|^2 |u|^2 dx \le \int_\Omega |\nabla_{\mathbb{G}} u|^2 dx.$$

Now if we assume that $\mathcal{A} = \nabla_{\mathbb{G}} \phi$ for some function ϕ, then (4.3) becomes

$$\mathcal{L}\phi + |\nabla_{\mathbb{G}} \phi|^2 = 0.$$

It follows that the function is harmonic (with respect to the sub-Laplacian $\mathcal{L}$).

$$w = e^{\phi} \ge 0$$

Then w is a constant > 0 or has a singularity. Let us consider

$$w := \sum_{k=1}^{m} \frac{1}{|(x a_k^{-1})'|^{N-2}},$$

and then take

$$\phi(x) = \ln(w).$$

Therefore

$$\begin{aligned}
\mathcal{A}(x) = \nabla_{\mathbb{G}}(\ln w) =& \frac{1}{w}\nabla_{\mathbb{G}}\left(\sum_{k=1}^{m}|(xa_k^{-1})'|^{2-N}\right)\\
=& \frac{1}{w}\sum_{k=1}^{m}\nabla_{\mathbb{G}}\left(\sum_{j=1}^{N}((xa_k^{-1})'_j)^2\right)^{\frac{2-N}{2}}\\
=& -\frac{N-2}{w}\left(\sum_{k=1}^{m}\frac{(xa_k^{-1})'}{|(xa_k^{-1})'|^N}\right),
\end{aligned}$$

and

$$|\mathcal{A}(x)|^2 = \sum_{j=1}^{N}|\mathcal{A}_j(x)|^2 = \left(\frac{N-2}{w}\right)^2\sum_{j=1}^{N}\left|\sum_{k=1}^{m}\frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N}\right|^2.$$

This completes the proof of Theorem 4.1. $\square$

We then also obtain the corresponding uncertainty principle.

Corollary 4.2. *Let $\Omega \subset \mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $N \geq 3$, $x = (x', x'') \in \mathbb{G}$ with $x' = (x'_1, \ldots, x'_N)$ being in the first stratum of $\mathbb{G}$. Let $a_k \in \mathbb{G}, k = 1, \ldots, m$, be the singularities. Then we have*

$$\frac{N-2}{2}\int_{\Omega}|u|^2dx \leq \left(\int_{\Omega}|\nabla_{\mathbb{G}}u|^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega}\frac{\left(\sum_{k=1}^{m}\frac{1}{|(xa_k^{-1})'|^{N-2}}\right)^2}{\sum_{j=1}^{N}\left|\sum_{k=1}^{m}\frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N}\right|^2}|u|^2dx\right)^{\frac{1}{2}}, \tag{4.4}$$

for all $u \in C_0^{\infty}(\Omega)$ and $1 < p_i < N$ for $i = 1, \ldots, N$.

Proof of Corollary 4.2. By (4.1) and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\int_{\Omega}|\nabla_{\mathbb{G}}u|^2dx\int_{\Omega}\frac{\left(\sum_{k=1}^{m}\frac{1}{|(xa_k^{-1})'|^{N-2}}\right)^2}{\sum_{j=1}^{N}\left|\sum_{k=1}^{m}\frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N}\right|^2}|u|^2dx\\
&\geq \left(\frac{N-2}{2}\right)^2\int_{\Omega}\frac{\sum_{j=1}^{N}\left|\sum_{k=1}^{m}\frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N}\right|^2}{\left(\sum_{k=1}^{m}\frac{1}{|(xa_k^{-1})'|^{N-2}}\right)^2}|u|^2dx\int_{\Omega}\frac{\left(\sum_{k=1}^{m}\frac{1}{|(xa_k^{-1})'|^{N-2}}\right)^2}{\sum_{j=1}^{N}\left|\sum_{k=1}^{m}\frac{(xa_k^{-1})'_j}{|(xa_k^{-1})'|^N}\right|^2}|u|^2dx\\
&\geq \left(\frac{N-2}{2}\right)^2\left(\int_{\Omega}|u|^2dx\right)^2.
\end{aligned}$$

The proof is complete. $\square$

5. Many-particle Hardy type inequality on $\mathbb{G}$

Suppose now that we have n particles, where n is a positive integer. Let $\mathbb{G}^n$ be the product $\mathbb{G}^n := \overbrace{\mathbb{G} \times \ldots \times \mathbb{G}}^{n}$. Now let us consider $x = (x_1, \ldots, x_n) \in \mathbb{G}^n$, $x_j \in \mathbb{G}$. Let $x \in \mathbb{G}^n$ with $x' = (x'_1, \ldots, x'_n)$ and $x'_i = (x'_{i1}, \ldots, x'_{iN})$ being the coordinates on the first stratum of $\mathbb{G}$ for $i = 1, \ldots, n$. The distance between particles $x_i, x_j \in \mathbb{G}$ can be defined by

$$r_{ij} := |(x_i x_j^{-1})'| = |x'_i - x'_j| = \sqrt{\sum_{k=1}^{N} (x'_{ik} - x'_{jk})^2}.$$

We also use the following notation

$$\nabla_{\mathbb{G}_i} = (X_{i1}, \ldots, X_{iN})$$

for the gradient associated to the i-th particle. We denote $\nabla_{\mathbb{G}^n} := (\nabla_{\mathbb{G}_1}, \ldots, \nabla_{\mathbb{G}_n})$, and

$$\mathcal{L}_i = \sum_{k=1}^{N} X_{ik}^2$$

is the sub-Laplacian associated to the i-th particle. We note that $\mathcal{L} = \sum_{i=1}^{N} \mathcal{L}_i$. We are new ready to prove the following crucial inequality in $\mathbb{R}^m$.

Lemma 5.1. *Let $m \geq 1$, and let*

$$\mathcal{A} = (\mathcal{A}_1(x), \ldots, \mathcal{A}_m(x))$$

be a mapping in $\mathcal{A} : \mathbb{R}^m \to \mathbb{R}^m$ whose components and their first derivatives are uniformly bounded in $\mathbb{R}^m$. Then for $u \in C_0^1(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \frac{1}{4} \frac{\left(\int_{\mathbb{R}^m} \operatorname{div}\mathcal{A}|u|^2 dx\right)^2}{\int_{\mathbb{R}^m} |\mathcal{A}|^2 |u|^2 dx}. \tag{5.1}$$

Proof of Lemma 5.1. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^m} \operatorname{div}\mathcal{A}|u|^2 dx\right| &= 2\left|\operatorname{Re}\int_{\mathbb{R}^m} \langle \mathcal{A}, \nabla u\rangle \overline{u} dx\right| \\ &\leq 2\left(\int_{\mathbb{R}^m} |\mathcal{A}|^2 |u|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^m} |\nabla u|^2 dx\right)^{1/2}.\end{aligned}$$

We have used the Cauchy-Schwarz inequality in the last line. The proof is finished by squaring this inequality. □

Theorem 5.2. *Let $\Omega \subset \mathbb{G}^n$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $N \geq 2$ and $n \geq 3$. Let $r_{ij} = |(x_i x_j^{-1})'| = |x'_i - x'_j|$. Then we have*

$$\int_{\Omega} |\nabla_{\mathbb{G}^n} u|^2 dx \geq \frac{(N-2)^2}{n} \int_{\Omega} \sum_{1 \leq i < j \leq n} \frac{|u|^2}{r_{ij}^2} dx, \tag{5.2}$$

for all $u \in C^1(\Omega)$.

The Euclidean case of inequality (5.2) is obtained by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, and J. Tidblom [6].

Proof of Theorem 5.2. Let us choose a mapping $\mathcal{B}_1$ in the following form

$$\mathcal{B}_1(x_i', x_j') := \frac{(x_i x_j^{-1})'}{r_{ij}^2}, \quad 1 \leq i < j \leq n.$$

And putting the mapping $\mathcal{B}_1$ in (5.1) we have

$$\begin{aligned}
\int_\Omega |(\nabla_{\mathbb{G}_i} - \nabla_{\mathbb{G}_j})u|^2 dx &\geq \frac{1}{4} \frac{\left(\int_\Omega \left((\mathrm{div}_{\mathbb{G}_i} - \mathrm{div}_{\mathbb{G}_j})\mathcal{B}_1\right)|u|^2 dx\right)^2}{\int_\Omega |\mathcal{B}_1|^2 |u|^2 dx} \\
&= \frac{1}{4} \frac{\left(\int_\Omega \frac{2(N-2)}{|(x_i x_j^{-1})'|^2} |u|^2 dx\right)^2}{\int_\Omega \frac{|u|^2}{|(x_i x_j^{-1})'|^2} dx} \\
&= (N-2)^2 \int_\Omega \frac{|u|^2}{r_{ij}^2} dx. \qquad (5.3)
\end{aligned}$$

Also, we introduce another mapping $\mathcal{B}_2$

$$\mathcal{B}_2(x) := \frac{\sum_{j=1}^n x_j'}{\left|\sum_{j=1}^n x_j'\right|^2},$$

and

$$\begin{aligned}
\mathrm{div}_{\mathbb{G}_i}\mathcal{B}_2 = \nabla_{\mathbb{G}_i} \cdot \mathcal{B}_2 &= \sum_{k=1}^N X_{ik} \left(\frac{\sum_{j=1}^n x_{jk}'}{|\sum_{j=1}^n x_j'|^2} \right) \\
&= \frac{Nn|\sum_{j=1}^n x_j'|^2 - 2n\left((\sum_{j=1}^n x_{j1}')^2 + \ldots + (\sum_{j=1}^n x_{jN}')^2\right)}{|\sum_{j=1}^n x_j'|^4} \\
&= \frac{Nn - 2n}{|\sum_{j=1}^n x_j'|^2}.
\end{aligned}$$

As before we put the mapping $\mathcal{B}_2$ in (5.1) and using above computation yielding

$$\begin{aligned}
\int_\Omega \left|\sum_{i=1}^n \nabla_{\mathbb{G}_i} u\right|^2 dx &\geq \frac{1}{4} \frac{\left(\int_\Omega (\sum_{i=1}^n \mathrm{div}_{\mathbb{G}_i}\mathcal{B}_2)|u|^2 dx\right)^2}{\int_\Omega |\mathcal{B}_2|^2 |u|^2 dx} \\
&= \frac{1}{4} \frac{\left(\int_\Omega \sum_{i=1}^n \frac{Nn-2n}{|\sum_{j=1}^n x_j'|^2} |u|^2 dx\right)^2}{\int_\Omega \frac{|u|^2}{\left|\sum_{j=1}^n x_j'\right|^2} dx} \\
&= \frac{(N-2)^2 n^4}{4} \int_\Omega \frac{|u|^2}{\left|\sum_{j=1}^n x_j'\right|^2} dx. \qquad (5.4)
\end{aligned}$$

Adding inequalities (5.3) and (5.4) and using the identity

$$n\sum_{i=1}^{n}|\nabla_{\mathbb{G}_i}u|^2=\sum_{1\leq i<j\leq n}\left|\nabla_{\mathbb{G}_i}u-\nabla_{\mathbb{G}_j}u\right|^2+\left|\sum_{i=1}^{n}\nabla_{\mathbb{G}_i}u\right|^2,$$

we arrive at

$$\sum_{i=1}^{n}\int_{\Omega}|\nabla_{\mathbb{G}_i}u|^2dx\geq\frac{(N-2)^2}{n}\int_{\Omega}\sum_{i<j}\frac{|u|^2}{r_{ij}^2}dx+\frac{(N-2)^2n^3}{4}\int_{\Omega}\frac{|u|^2}{\left|\sum_{j=1}^{n}x'_j\right|^2}dx. \tag{5.5}$$

Because the last term on right-hand side is positive, we get

$$\sum_{i=1}^{n}\int_{\Omega}|\nabla_{\mathbb{G}_i}u|^2dx\geq\frac{(N-2)^2}{n}\int_{\Omega}\sum_{i<j}\frac{|u|^2}{r_{ij}^2}dx.$$

Also we have

$$\begin{aligned}\sum_{i=1}^{n}|\nabla_{\mathbb{G}_i}u|^2&=(\nabla_{\mathbb{G}_1}u)^2+\ldots+(\nabla_{\mathbb{G}_n}u)^2\\&=|(\nabla_{\mathbb{G}_1}u,\ldots,\nabla_{\mathbb{G}_n}u)|^2\\&=|\nabla_{\mathbb{G}^n}u|^2.\end{aligned}$$

The proof of Theorem 5.2 is complete. $\square$

The following theorem deals with the total separation of $n\geq 2$ particles.

Theorem 5.3. *Let $\Omega\subset\mathbb{G}^n$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $\rho^2:=\sum_{i<j}|(x_ix_j^{-1})'|^2=\sum_{i<j}|x'_i-x'_j|^2$ with $x'_i\neq x'_j$. Then we have*

$$\int_{\Omega}|\nabla_{\mathbb{G}}u|^2dx=n\left(\frac{(n-1)}{2}N-1\right)^2\int_{\Omega}\frac{|u|^2}{\rho^2}dx+\int_{\Omega}|\nabla_{\mathbb{G}}\rho^{-2\alpha}u|^2\rho^{4\alpha}dx \tag{5.6}$$

for all $u\in C_0^\infty(\Omega)$ with $\alpha=\frac{2-(n-1)N}{4}$.

The Euclidean case of inequality (5.6) was obtained by Douglas Lundholm [9].

Proposition 5.4. *Let $\Omega\subset\mathbb{G}^n$ be an open set, where $\mathbb{G}$ is a stratified group with N being the dimension of the first stratum. Let $f:\Omega\to(0,\infty)$ be twice differentiable. Then for any function $u\in C_0^\infty(\Omega)$ and $\alpha\in\mathbb{R}$, we have*

$$\int_{\Omega}|\nabla_{\mathbb{G}}u|^2dx=\int_{\Omega}\left(\alpha(1-\alpha)\frac{|\nabla_{\mathbb{G}}f|^2}{f^2}-\alpha\frac{\mathcal{L}f}{f}\right)|u|^2dx+\int_{\Omega}|\nabla_{\mathbb{G}}v|^2f^{2\alpha}dx, \tag{5.7}$$

where $v:=f^{-\alpha}u$.

Proof of Proposition 5.4. Let us compute for $u=f^\alpha v$, that

$$\nabla_{\mathbb{G}}u=\alpha f^{\alpha-1}(\nabla_{\mathbb{G}}f)v+f^\alpha\nabla_{\mathbb{G}}v.$$

Then by squaring the above expression we have

$$\begin{aligned}|\nabla_{\mathbb{G}}u|^2 &= \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}}f|^2|v|^2 + \mathrm{Re}(2\alpha v f^{2\alpha-1}(\nabla_{\mathbb{G}}f)\cdot(\nabla_{\mathbb{G}}v)) + f^{2\alpha}|\nabla_{\mathbb{G}}v|^2\\ &= \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}}f|^2|v|^2 + \alpha f^{2\alpha-1}(\nabla_{\mathbb{G}}f)\cdot\nabla_{\mathbb{G}}|v|^2 + f^{2\alpha}|\nabla_{\mathbb{G}}v|^2.\end{aligned}$$

By integrating this expression over Ω, we have

$$\begin{aligned}\int_\Omega |\nabla_{\mathbb{G}}u|^2dx &= \int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}}f|^2|v|^2dx\\ &\quad+\int_\Omega \mathrm{Re}(\alpha f^{2\alpha-1}(\nabla_{\mathbb{G}}f)\cdot\nabla_{\mathbb{G}}|v|^2)dx + \int_\Omega f^{2\alpha}|\nabla_{\mathbb{G}}v|^2dx\\ &= \int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}}f|^2|v|^2dx\\ &\quad-\alpha\int_\Omega \nabla_{\mathbb{G}}\cdot(f^{2\alpha-1}\nabla_{\mathbb{G}}f)|v|^2dx + \int_\Omega f^{2\alpha}|\nabla_{\mathbb{G}}v|^2dx.\end{aligned}$$

We have used the integration by parts to the middle term on the right-hand side. Then

$$\nabla_{\mathbb{G}}\cdot(f^{2\alpha-1}\nabla_{\mathbb{G}}f) = (2\alpha-1)f^{2\alpha-2}|\nabla_{\mathbb{G}}f|^2 + f^{2\alpha-1}\mathcal{L}f,$$

and by using this fact we get

$$\begin{aligned}\int_\Omega |\nabla_{\mathbb{G}}u|^2dx &= \int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla_{\mathbb{G}}f|^2|v|^2dx - \int_\Omega \alpha f^{2\alpha-1}\mathcal{L}f|v|^2dx\\ &\quad-\int_\Omega \alpha(2\alpha-1)f^{2\alpha-2}|\nabla_{\mathbb{G}}f|^2|v|^2dx + \int_\Omega f^{2\alpha}|\nabla_{\mathbb{G}}v|^2dx.\end{aligned}$$

Putting back $v = f^{-\alpha}u$ and collecting the terms we arrive at (5.7). ☐

Proof of Theorem 5.3. The following computation gives

$$\nabla_{\mathbb{G}_k}\rho^2 = (X_{k1}\rho^2,\ldots,X_{kN}\rho^2) = 2\sum_{k\neq j}^{n}(x_k x_j^{-1})',$$

where $\nabla_{\mathbb{G}_k} = (X_{k1},\ldots,X_{kN})$. Hence

$$\mathcal{L}\rho^2 = 2\sum_{k=1}^{n}\sum_{k\neq j}^{n}\nabla_{\mathbb{G}_k}\cdot(x_k x_j^{-1})' = 2n(n-1)N, \tag{5.8}$$

$$|\nabla_{\mathbb{G}}\rho^2|^2 = 8\sum_{1\leq i<j\leq n}|(x_k x_j^{-1})'|^2 + 8\sum_{k=1}^{n}\sum_{1\leq i<j\leq n}(x_k x_i^{-1})'\cdot(x_k x_j^{-1})' = 4n\rho^2, \tag{5.9}$$

where in the last step we used the identity

$$\sum_{k=1}^{n}\sum_{1\leq i<j\leq n}(x_k x_i^{-1})'\cdot(x_k x_j^{-1})' = \frac{n-2}{2}\sum_{1\leq i<j\leq n}|(x_i x_j^{-1})'|^2.$$

By putting (5.8) and (5.9) in Proposition 5.4 with $f=\rho^2$ we have

$$\int_\Omega |\nabla_{\mathbb{G}}u|^2dx = 4n\alpha\left(\frac{2-(n-1)N}{2}-\alpha\right)\int_\Omega \frac{|u|^2}{\rho^2}dx + \int_\Omega |\nabla_{\mathbb{G}}\rho^{-2\alpha}u|^2\rho^{4\alpha}dx.$$

To optimise we differentiate the integral

$$4n\alpha\left(\frac{2-(n-1)N}{2}-\alpha\right)\int_{\Omega}\frac{|u|^2}{\rho^2}dx$$

with respect to α, then we have

$$\frac{2-(n-1)N}{2}-2\alpha=0,$$

and

$$\alpha=\frac{2-(n-1)N}{4},$$

which completes the proof of Theorem 5.3. □

6. **Hardy type inequalities with exponential weights on $\mathbb{G}$**

In this section, we get the horizontal Hardy inequality with exponential weights on $\mathbb{G}$.

Theorem 6.1. *Let $\Omega\subset\mathbb{G}$ be an open set, where $\mathbb{G}$ is a stratified group with $N\geq 3$ being the dimension of the first stratum. Let $x_0\in\Omega$. Then we have*

$$\int_{\Omega}e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}\left(\frac{(N-2)^2}{4|x'|^2}-\frac{N}{4\alpha}+\frac{|(xx_0^{-1})'|^2}{16\lambda^2}\right)|u|^2dx\leq\int_{\Omega}e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|\nabla_{\mathbb{G}}u|^2dx \tag{6.1}$$

for all $u\in C^1(\Omega)$ and for each $\lambda>0$.

In the Euclidean case, this inequality is called two parabolic-type Hardy inequality, which was obtained by Zhang [18].

Proof of Theorem 6.1. Let us recall the horizontal Hardy inequality [13] for all $v\in C^1(\Omega)$,

$$\frac{(N-2)^2}{4}\int_{\Omega}\frac{|v|^2}{|x'|^2}dx\leq\int_{\Omega}|\nabla_{\mathbb{G}}v|^2dx. \tag{6.2}$$

Let $v=e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}u$, then

$$\nabla_{\mathbb{G}}v=e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}\nabla_{\mathbb{G}}u-\frac{(xx_0^{-1})'}{4\lambda}e^{-\frac{|(xx_0^{-1})'|^2}{8\lambda}}u,$$

for all $v\in C^1(\Omega)$. Then by inequality (6.2) we have

$$\frac{(N-2)^2}{4}\int_{\Omega}e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}\frac{|u|^2}{|x'|^2}dx\leq\int_{\Omega}e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|\nabla_{\mathbb{G}}u|^2+\frac{|(xx_0^{-1})'|^2}{16\lambda^2}e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}|u|^2dx \tag{6.3}$$

$$-\operatorname{Re}\frac{1}{2\lambda}\int_{\Omega}(xx_0^{-1})'\cdot(\nabla_{\mathbb{G}}u)ue^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}}dx.$$

By the integration by parts in the last term of right-hand side of the inequality we have

$$\mathrm{Re}\int_\Omega (xx_0^{-1})' \cdot (\nabla_{\mathbb{G}} u) u e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}} dx = -\frac{1}{2}\int_\Omega \left(N - \frac{|(xx_0^{-1})'|^2}{2\lambda}\right) e^{-\frac{|(xx_0^{-1})'|^2}{4\lambda}} |u|^2 dx. \tag{6.4}$$

By putting equality (6.4) in (6.3) and rearranging it, we prove Theorem 6.1. □

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